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TWISTED ACTION OF THE SYMMETRIC GROUP ON THE COHOMOLOGY OF A FLAG MANIFOLD

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Abstract. Classes dual to Schubert cycles constitute a basis on the cohomology ring of the flag manifold $\mathcal{F}$, self-adjoint up to indexation with respect to the intersection form. Here, we study the bilinear form

$$(X, Y) := \langle X \cdot Y,\, c(\mathcal{F})\rangle$$

where X, Y are cocycles, $c(\mathcal{F})$ is the total Chern class of $\mathcal{F}$ and $\langle\ ,\ \rangle$ is the intersection form. This form is related to a twisted action of the symmetric group of the cohomology ring, and to the degenerate affine Hecke algebra. We give a distinguished basis for this form, which is a deformation of the usual basis of Schubert polynomials, and apply it to the computation of the Schubert cycle expansions of Chern classes of flag manifolds.

1. Introduction and preliminaries. Let V be a complex vector space of dimension n, and $\mathcal{F} = \mathcal{F}(V)$ be the variety of complete flags in V. It is well known that the cohomology ring $H^*(\mathcal{F}, \mathbb{C})$ is the quotient of the polynomial ring $\mathbb{C}[X] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ by the ideal $\mathcal{I}^+$ of symmetric polynomials without constant term.

Let σ_i, $i = 1, \ldots, n-1$ be the simple transposition exchanging x_i and x_{i+1}. Denote

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by ∂_i the linear operator on $\mathbb{C}[x_1,\ldots,x_n]$ defined by

$$\partial_i f := \frac{f-\sigma_i f}{x_i - x_{i+1}} \tag{1}$$

(Newton's divided difference). The operators $\partial_1,\ldots,\partial_{n-1}$ induce operators on $H^*(\mathcal{F})$.

According to [1] and [4], the basis of Schubert cycles can be obtained from the class of a point $P = \frac{1}{n!}\prod_{i<j}(x_i - x_j)$ by successive applications of divided difference operators. Taking as representative of P the polynomial $X := x_1^{n-1}x_2^{n-2}\cdots x_1^0$, one obtains polynomials X_μ, $\mu \in \mathfrak{S}_n$, called Schubert polynomials, which represent the Schubert subvarieties in the cohomology ring [11]. A detailed account of the algebraic theory of Schubert polynomials can be found in Macdonald's treatise [14].

Divided differences satisfy the braid relations

$$\begin{cases} \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1} \\ \partial_i\partial_j &= \partial_j\partial_i \qquad \text{for} \quad |i-j|>1, \end{cases} \tag{2}$$

but the squares ∂_i^2 are null. These relations allow to define operators ∂_μ for any permutation $\mu \in \mathfrak{S}_n$: if $\mu = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$ is a reduced decomposition of μ, one sets $\partial_\mu = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_m}$. The result does not depend on the choice of a particular reduced decomposition of μ.

To recover an action of the symmetric group, one can take any $q \in \mathbb{C}$ and define

$$D_i := \sigma_i + q\partial_i, \qquad 1 \le i \le n-1. \tag{3}$$

These operators still satisfy the braid relations

$$\begin{cases} D_iD_{i+1}D_i &= D_{i+1}D_iD_{i+1} \\ D_iD_j &= D_jD_i \qquad (|i-j|>1) \end{cases} \tag{4}$$

together with

$$D_i^2 = 1, \tag{5}$$

so that they generate a representation of the symmetric group $\mathfrak{S}_n$ on the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$, as well as on the cohomology ring $H^*(\mathcal{F},\mathbb{C})$. These operators have been considered by Cherednik and Bernstein (cf. [2], [3]). Similar operators, acting on the equivariant K-theory of flag manifolds, have been used by Lusztig [13]. More general operators satisfying braid relations have been given in [12].

As ∂_i decreases degrees by 1, all $q \neq 0$ will give equivalent representations of $\mathfrak{S}_n$, and by homogeneity, the general case can be recovered from the case $q = 1$. For simplicity, we set $q = 1$, and write

$$s_i := \sigma_i + \partial_i. \tag{6}$$

We denote as above by s_μ the product of operators s_i corresponding to a permutation μ. Remark that the operator algebra generated by the s_i and the variables x_j (interpreted as operators $f \mapsto x_j f$) is isomorphic to the degenerate affine Hecke algebra considered in [2].

Schubert calculus for other classical groups can be found in the work of Fulton [7] and of Pragacz and Ratajski [15].

This paper is organized as follows. We first define certain elements (Yang-Baxter operators) of the degenerate affine Hecke algebra. Then we use them to define a bilinear form on the cohomology of a flag manifold. We exhibit a distinguished basis, called *affine*

Schubert polynomials, and compute its adjoint basis. We then apply this formalism to the computation of the Schubert expansions of Chern classes.

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2. Yang-Baxter operators. We shall define inductively operators $\Box_\mu$ and ∇_μ associated with any permutation μ in $\mathfrak{S}_n$. Set $\Box_{12\ldots n} = 1$, $\nabla_{12\ldots n} = 1$, and, if $\mu = \sigma_i \alpha$ with $\ell(\mu) = \ell(\alpha) + 1$, and $\beta = \alpha^{-1}$,

$$\begin{cases} \Box_\mu = \left(s_i + \frac{1}{\beta_{i+1} - \beta_i}\right) \Box_\alpha \\ \nabla_\mu = \left(s_i - \frac{1}{\beta_{i+1} - \beta_i}\right) \nabla_\alpha \end{cases} \tag{7}$$

Using the braid relations (4), one can check that this definition is consistent, i.e. does not depend on the chosen factorization (see [2, 3] and [6]). This follows in fact from a classical solution of the Yang-Baxter equation. In [17], C. N. Yang observed that the operators defined by $Y_i(u) = u^{-1} + \sigma_i$, where u is a scalar parameter and σ_i the transposition $(i, i+1)$ satisfy the "Quantum Yang-Baxter Equation with spectral parameter":

$$Y_i(u-v)Y_{i+1}(u-w)Y_i(v-w) = Y_{i+1}(v-w)Y_i(u-w)Y_{i+1}(u-v) \tag{8}$$

It follows that given a n-tuple of parameters $\mathbf{u} = (u_1, \ldots, u_n)$, one can define for any permutation $\mu \in \mathfrak{S}_n$ an operator $R_\mu(\mathbf{u})$ by the following prescription: $Y_\mu(\mathbf{u}) = Y_i(u_{\beta(i+1)} - u_{\beta(i)})R_\alpha(\mathbf{u})$, where, as above, $R_{12\ldots n} = 1$, $\mu = \sigma_i\alpha$, $\ell(\mu) = \ell(\alpha) + 1$ and $\beta = \alpha^{-1}$. Then, our operators (7) are respectively $R_\mu(\mathbf{u})$ and $R_\mu(-\mathbf{u})$, where $\mathbf{u} = (1, 2, \ldots, n)$ and σ_i is interpreted as s_i.

For the maximal element $\omega = (n, n-1, \ldots, 1)$ of $\mathfrak{S}_n$, one has the following factorization property (given in [6] for the case of the Hecke algebra):

PROPOSITION 2.1. *Define* $\theta = \prod_{1 \le i < j \le n} (1 + x_i - x_j)$ *and* $\theta^* = \prod_{1 \le i < j \le n} (1 - x_i + x_j)$. *Then, for any polynomial* f,

(i) $\nabla_\omega f = \theta^* \, \partial_\omega f$

(ii) $\Box_\omega f = \partial_\omega(\theta f)$.

Proof. Recall that the classes of the Schubert polynomials X_μ, $\mu \in \mathfrak{S}_n$, form a basis of $H^*(\mathcal{F}) = \mathbb{C}[X]/\mathcal{I}^+$. Given μ and i such that $\ell(\mu\sigma_i) > \ell(\mu)$, the polynomial X_μ is symmetrical in x_i and x_{i+1}. As such, it is sent to 0 by the operator $\nabla_{\sigma_i} = \sigma_i + \partial_i - 1$.

Now, for any permutation $\mu \neq \omega$, there exists an i such that $\ell(\mu\sigma_i) > \ell(\mu)$. If we choose a reduced decomposition of ω ending by σ_i, $\omega = \nu\sigma_i$, say, we see that X_μ is sent to 0 by $\partial_\omega = \partial_\nu \partial_{\sigma_i}$ and by $\nabla_\omega = \nabla_\mu \nabla_{\sigma_i}$.

Thus, ∇_ω as well as ∂_ω annihilate all Schubert polynomials X_μ for $\mu \neq \omega$. Finally, $X_\omega = x_1^{n-1} \ldots x_n^0$ is sent to 1 by ∂_ω. To conclude, it remains to prove that

$$\nabla_\omega(X_\omega) = \prod_{1 \le i < j \le n} (1 - x_i + x_j).$$

This formula can be proved by induction on n using the factorization

$$\omega_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \omega_{n-1},$$

which gives

$$\nabla_{\omega_n} = \left(s_1 - 1\right) \cdots \left(s_{n-1} - \frac{1}{n-1}\right) \nabla_{\omega_{n-1}}. \blacksquare$$

3. Quadratic form. Recall that the intersection form of the cohomology ring $H^*(\mathcal{F}, \mathbb{C})$ is induced by the form on $\mathbb{C}[X]$

$$\langle f, g \rangle = \partial_\omega (fg)|_0 = \partial_\omega (fg|_{\ell(\omega)}) \tag{9}$$

where $f|_k$ denotes the homogeneous component of degree k of f (cf. [1], [5]). With respect to this form, the Schubert polynomials satisfy

$$\langle X_\mu, X_\nu \rangle = \begin{cases} 1 & \text{if } \nu = \omega\mu \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The tangent bundle $T\mathcal{F}$ of the flag manifold has a composition sequence $\{L_i L_j^{-1}\}_{i<j}$ where $L_1, L_2, \ldots, L_n$ are the tautological line bundles on $\mathcal{F}$. The total Chern class of L_i being $c(L_i) = 1 + x_i$, the total Chern class of the tangent bundle of $\mathcal{F}$ is

$$c(\mathcal{F}) = \prod_{i<j} (1 + x_i - x_j) \tag{11}$$

(see e.g. [8], our convention is $L_i = \xi_i^*$ in the notation of [8]). Consider now the following quadratic form on $\mathbb{C}[X]$:

Definition 3.1.

$$(f, g) := \Box_\omega (fg)|_0.$$

Thus, in the cohomology ring, we see from Proposition 2.1 that

$$(f, g) = \langle f, g\, c(\mathcal{F}) \rangle = \langle f\, c(\mathcal{F}), g \rangle. \tag{12}$$

Lemma 3.2. *The operators $\Box_i$ are self-adjoint with respect to the quadratic form (,).*

Proof. For any i, $\Box_\omega \Box_i = 2\Box_\omega$, since $\Box_i^2 = 2\Box_i$ and since one can find a reduced decomposition of ω ending with σ_i. Now,

$$(\Box_i f, g) = \Box_\omega \left((\Box_i f) g \right) |_0 = \frac{1}{2} \Box_\omega \Box_i \left((\Box_i f) g \right) |_0 = \frac{1}{2} \Box_\omega \left((\Box_i f)(\Box_i g) \right) |_0$$

since $\Box_i f$ is a scalar for $\Box_i$, being symmetrical in x_i, x_{i+1}. The last expression being symmetrical in f, g, this proves that $(\Box_i f, g) = (f, \Box_i g)$. $\blacksquare$

4. Affine Schubert polynomials. Let $\mathcal{H}_n$ be the linear subspace of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the monomials $x^I = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ such that $i_k \leq n - k$. Let Π be the projector from $\mathbb{C}[X]$ onto $\mathcal{H}_n$ associating to a polynomial P the unique representative in $\mathcal{H}_n$ of its class $\overline{P} \in \mathbb{C}[X]/\mathcal{I}^+$.

Definition 4.1. Let $\mu \in \mathfrak{S}_n$. The *affine Schubert polynomial* of index μ is defined by

$$Z_\mu = \Pi(\Box_{\mu^{-1}\omega} Z_\omega)$$

where $Z_\omega := X_\omega = x_1^{n-1} x_2^{n-2} \cdots x_n^0$.

EXAMPLE 4.2. For $n = 3$,

$$\begin{aligned} Z_{321} &= x_1^2 x_2 \\ Z_{312} &= x_1^2 \\ Z_{231} &= x_1 x_2 \\ Z_{213} &= x_1 - 1/2 x_1 x_2 - x_1^2 \\ Z_{132} &= x_1 + x_2 - x_1 x_2 - 1/2 x_1^2 \\ Z_{123} &= 1 \end{aligned}$$

In general, one has $Z_\omega = X_\omega$, $Z_\mu = X_\mu$+(terms of degree $> \ell(\mu)$), and $Z_{id} = 1$, the last identity being due to the fact that $\square_\omega(Z_\omega)$ is symmetrical with term of lowest degree $X_{id} = 1$.

EXAMPLE 4.3. For $n = 3$,

$$\begin{aligned} Z_{321} &= X_{321} \\ Z_{312} &= X_{312} \\ Z_{231} &= X_{231} \\ Z_{213} &= X_{213} - 1/2 X_{231} - X_{312} \\ Z_{132} &= X_{132} - X_{231} - 1/2 X_{312} \\ Z_{123} &= X_{123} \end{aligned}$$

THEOREM 4.4. *The polynomials Z_μ, $\mu \in \mathfrak{S}_n$, form a basis of $\mathcal{H}_n$. The quadratic form $(\ ,\ )$ is positive definite, and the adjoint basis of $\{Z_\mu\}$ is $\{Z_\mu^\vee\}$ where $Z_\mu^\vee = \Pi(\nabla_{\mu^{-1}\omega} X_\omega)$.*

Proof. Z_μ is a non-homogeneous polynomial with the Schubert polynomial X_μ as its term of smallest degree. Since the classes of the Schubert polynomials form a basis of $H^*(\mathcal{F})$, the same is true for the Z_μ.

The polynomials $Z_\omega^\vee Z_\mu \theta$ (for $\mu \neq id$) have no component of degree $\ell(\omega)$. Therefore, their images under ∂_ω are symmetric polynomials without constant term, which proves that for all $\mu \neq id$, $(Z_\omega^\vee, Z_\mu) = 0$. On the other hand,

$$(Z_\omega^\vee, Z_{12\ldots n}) = (Z_\omega^\vee, 1) = \partial_\omega (X_\omega \theta)\,|_0 = \partial_\omega \left((X_\omega\theta)|_{\ell(\omega)}\right) = \partial_\omega X_\omega = X_{12\ldots n} = 1.$$

For the general case of a $Z_\nu^\vee$, one uses induction on the length of ν. Let ν and i be such that $\ell(\nu\sigma_i) < \ell(\nu)$. Then, for any μ and an appropriate constant k

$$(Z_\mu, Z_\nu^\vee) = (Z_\mu, (s_i - k) Z_\nu^\vee) = ((s_i - k) Z_\mu, Z_\nu^\vee) = k'(Z_\mu, Z_\nu^\vee) + k''(Z_{\mu\sigma_i}, Z_\nu^\vee)$$

(for some other scalars k', k''). By induction, one can suppose $(Z_\mu, Z_\nu^\vee) = 0$ for $\mu\nu^{-1} \neq \omega$.

One is thus reduced to study the case

$$\mu\sigma_i\nu^{-1} = \omega, \quad \ell(\mu\sigma_i) > \ell(\mu).$$

In that case,

$$Z_{\mu\sigma_i} = \left(s_i + \frac{1}{r}\right) Z_\mu \quad \text{and} \quad Z_{\mu\sigma_i}^\vee = \left(s_i - \frac{1}{r}\right) Z_\mu^\vee$$

for a certain integer r. Then, we check

$$(Z_{\mu\sigma_i}, Z_{\nu\sigma_i}^\vee) = \left(\left(s_i + \frac{1}{r}\right) Z_\mu\,,\, \left(s_i - \frac{1}{r}\right) Z_\mu^\vee\right) = \left(\left(s_i^2 - \frac{1}{r^2}\right) Z_\mu\,,\, Z_\nu^\vee\right) = 0,$$

and

$$(Z_\mu, Z^\vee_{\nu\sigma_i}) = \left(\left(s_i + \frac{1}{r} - \frac{2}{r}\right) Z_\mu \,,\, Z^\vee_\nu\right) = (Z_{\mu\sigma_i}, Z^\vee_\nu) - \frac{2}{r}(Z_\mu, Z^\vee_\nu) = 1 - 0. \ \blacksquare$$

EXAMPLE 4.5. Again for $n = 3$,

$$\begin{aligned}
Z^\vee_{321} &= x_1^2 x_2 \\
Z^\vee_{312} &= x_1^2 - 2x_1^2 x_2 \\
Z^\vee_{231} &= x_1 x_2 - 2x_1^2 x_2 \\
Z^\vee_{213} &= x_1 - 3/2 x_1 x_2 - 3x_1^2 + 3x_1^2 x_2 \\
Z^\vee_{132} &= x_1 + x_2 - 3x_1 x_2 - 3/2 x_1^2 + 3x_1^2 x_2 \\
Z^\vee_{123} &= 1 - 4x_1 - 2x_2 + 6x_1 x_2 + 6x_1^2 - 6x_1^2 x_2
\end{aligned}$$

5. Change of basis. The operators ∂_i are self-adjoint with respect to $\langle\ ,\ \rangle$, but σ_i is adjoint to $-\sigma_i$. This implies that $-s_i$ is adjoint to $\bar{s}_i := \sigma_i - \partial_i$.

Let us define $\overline{\Box}_\mu$, $\overline{\nabla}_\mu$ to be the images of $\Box_\mu$ and ∇_μ under the replacement $s_i \mapsto \bar{s}_i$. We also define

$$\overline{Z}_\mu := (-1)^{\ell(\omega\mu)}\Pi\left(\overline{\Box}_{\mu^{-1}\omega} Z_\omega\right), \qquad \overline{Z}^\vee_\mu := (-1)^{\ell(\omega\mu)}\Pi\left(\overline{\nabla}_{\mu^{-1}\omega} Z_\omega\right).$$

Then, $(-1)^{\ell(\mu)}\overline{Z}_\mu$ is obtained from Z_μ under the transformation $x_i \mapsto -x_i$, since signs in the expansion of Z_μ correspond to the degree.

LEMMA 5.1. *$\{\overline{Z}_{\omega\mu}\}$ is the adjoint basis of $\{Z_\mu\}$ with respect to $\langle\ ,\ \rangle$,* i.e. *one has $\langle\overline{Z}_{\omega\mu}, Z_\mu\rangle = 1$ and $\langle\overline{Z}_{\omega\mu}, Z_\nu\rangle = 0$ for $\nu \neq \mu$.*

Similarly, $\{\overline{Z}^\vee_{\omega\mu}\}$ is the adjoint basis of $\{Z^\vee_\mu\}$ for $\langle\ ,\ \rangle$.

Proof. As in Section 4, the lemma is proved by induction on the length of μ, starting from the case

$$\langle Z_{\omega\mu}, \overline{Z}_\omega\rangle = 0 \quad \text{if} \quad \mu \neq \omega.$$

Take i such that $\ell(\mu\sigma_i) > \ell(\mu)$. Then,

$$\langle Z_{\omega\mu\sigma_i}, \overline{Z}_\nu\rangle = \left\langle\left(s_i + \frac{1}{r}\right) Z_{\omega\mu} \,,\, \overline{Z}_\nu\right\rangle = \left\langle Z_{\omega\mu} \,,\, \left(-\bar{s}_i + \frac{1}{r}\right)\overline{Z}_\nu\right\rangle.$$

Since $(-\bar{s}_i + \frac{1}{r})\overline{Z}_\nu$ is a linear combination of $\overline{Z}_\nu$ and $\overline{Z}_{\nu\sigma_i}$, the nullity of the scalar products $\langle Z_{\omega\mu\sigma_i}, \overline{Z}_\nu\rangle$ follows from those of $\langle Z_{\omega\mu}, Z_\nu\rangle$ for $\nu \neq \mu$ and $\nu \neq \mu\sigma_i$. In the special case $\nu = \mu\sigma_i$, one has

$$\langle Z_{\omega\mu\sigma_i}, \overline{Z}_\mu\rangle = \left\langle Z_{\omega\mu} \,,\, \left(-\bar{s}_i + \frac{1}{r}\right)\left(-\bar{s}_i - \frac{1}{r}\right) Z_{\mu\sigma_i}\right\rangle = \left\langle Z_{\omega\mu} \,,\, \left(1 - \frac{1}{r^2}\right) Z_{\mu\sigma_i}\right\rangle$$

which is null. $\blacksquare$

EXAMPLE 5.2.

$$\begin{aligned}
\langle Z_{23514}, \overline{Z}_{41352}\rangle &= \left\langle\left(s_2 + \frac{1}{2}\right) Z_{2\mathbf{53}14} \,,\, \left(-\bar{s}_2 - \frac{1}{2}\right)\overline{Z}_{4\mathbf{31}52}\right\rangle \\
&= \left\langle\left(s_2 - \frac{1}{2}\right)\left(s_2 + \frac{1}{2}\right) Z_{25314} \,,\, \overline{Z}_{43152}\right\rangle \\
&= \left\langle\left(1 - \frac{1}{4}\right) Z_{25314} \,,\, \overline{Z}_{43152}\right\rangle = 0.
\end{aligned}$$

Let $\{A_\mu\}$ and $\{B_\nu\}$ be two bases of $\mathcal{H}_n$. We denote by $M(A,B)$ the transition matrix from the basis $\{A_\mu\}$ to the basis $\{B_\nu\}$, with the convention

$$A_\mu = \sum_\nu M(A,B)_{\mu\nu} B_\nu. \tag{13}$$

For example, $M(Z,X)_{\mu\nu} = \langle Z_\mu, X_{\omega\nu}\rangle$ and $M(X,Z)_{\mu\nu} = (X_\mu, Z^\vee_{\omega\nu})$. These matrices have a symmetry property, thanks to the following property of the scalar product:

$$\langle \omega P, \omega Q\rangle = (-1)^{\ell(\omega)}\langle P, Q\rangle. \tag{14}$$

Indeed, taking into account the two identities

$$\omega(X_\mu) = (-1)^{\ell(\mu)} X_{\omega\mu\omega}, \quad \omega(Z_\mu) = (-1)^{\ell(\mu)}\overline{Z}_{\omega\mu\omega} \tag{15}$$

we see that the four matrices

$$M(Z,X),\ M(X,Z),\ M(Z^\vee,X) \text{ and } M(X,Z^\vee)$$

possess the symmetry

$$M_{\mu\nu} = M_{\omega\mu\omega,\omega\nu\omega}. \tag{16}$$

Furthermore, we have the following relation between these matrices and their inverses:

Theorem 5.3. *The inverse of $M(Z,X)$ is a matrix with nonnegative entries, given by*

$$M(X,Z)_{\mu\nu} = |M(Z,X)_{\nu\omega,\mu\omega}|.$$

Similarly,

$$M(X,Z^\vee)_{\mu\nu} = |M(Z^\vee,X)_{\nu\omega,\mu\omega}|,$$

where $|\cdot|$ denotes the absolute value.

Proof. The first matrix corresponds to the expansions

$$Z_\mu = \sum_\nu \langle Z_\mu, X_{\omega\nu}\rangle X_\nu.$$

The inverse formulas are, according to Lemma 5.1,

$$X_\nu = \sum_\mu \langle \overline{Z}_{\omega\mu}, X_\nu\rangle Z_\mu.$$

But now, $\langle \overline{Z}_{\omega\mu}, X_\nu\rangle = |\langle Z_{\omega\mu}, X_\nu\rangle|$, whence the first part of the theorem follows. The proof of the second part is similar. ∎

Thus, the inverse of the matrix $M(Z,X)$ is obtained from $M(Z,X)$ by reflection through the antidiagonal $(\mu,\nu) \longrightarrow (\omega\nu,\omega\mu)$ and suppression of the signs.

Corollary 5.4. *For any pair of permutations,*

$$(X_\mu, Z^\vee_\eta) = |\langle X_{\omega\mu\omega}, Z_{\omega\eta\omega}\rangle|$$

and

$$(X_\mu, Z_\eta) = |\langle X_{\omega\mu\omega}, Z^\vee_{\omega\eta\omega}\rangle|.$$

Indeed,

$$X_\mu = \sum_\eta (X_\mu, Z^\vee_\eta) Z_{\omega\eta}$$

but we have just seen that the coefficients of the expansion of X_μ in the basis $Z_{\omega\eta}$ are the $\langle \overline{Z}_\eta, X_\mu\rangle$. ∎

6. Schubert expansions of Chern classes. Let $|I|$ be a composition of n, i.e. $I \in \mathbb{N}^r$ with $|I| := i_1 + \ldots + i_r = n$, and let $J_1, \ldots, J_r$ be the associated decomposition of the interval $[1, n]$, that is

$$J_1 = [1, i_1], \quad J_2 = [i_1 + 1, i_1 + i_2], \quad \ldots, \quad J_r = [i_1 + \cdots + i_{r-1} + 1, n].$$

Let $\mathcal{F}_I$ be the variety of flags

$$V_0 = \{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_r = V$$

such that $\dim V_k = i_1 + \cdots + i_k$. Let also $\mathfrak{S}_I$ denote the Young subgroup

$$\mathfrak{S}(J_1) \times \mathfrak{S}(J_2) \times \cdots \times \mathfrak{S}(J_r) \ \subset \mathfrak{S}_n$$

associated to the composition I. The Chern class θ^I of the tangent bundle of $\mathcal{F}_I$ is the maximal $\mathfrak{S}_I$-invariant factor of θ:

$$\theta^I = \theta / (\theta_{J_1} \theta_{J_2} \cdots \theta_{J_r})$$

where

$$\theta_{J_k} = \prod_{\substack{i<j \\ i,j \in J_k}} (1 + x_i - x_j).$$

A basis of the cohomology ring $H^*(\mathcal{F}_I)$ is the set of Schubert polynomials X_μ with μ minimal in its right coset $\mu \mathfrak{S}_I$. In other words, one restricts the Schubert basis (X_μ) to those μ such that $\mu_1 < \ldots < \mu_{i_1}$, $\mu_{i_1+1} < \ldots < \mu_{i_1+i_2}$, …, $\mu_{i_1+\ldots+i_{r-1}+1} < \ldots < \mu_n$. Since $\mu_i < \mu_{i+1}$ iff X_μ is symmetrical in x_i and x_{i+1}, the Schubert basis of $H^*(\mathcal{F}_I)$ consists of those Schubert polynomials which are invariant under $\mathfrak{S}_I$.

Define the *Chern coefficient* $c_\mu[\mathcal{F}_I]$ of the variety $\mathcal{F}_I$ as the coefficient of (the class) of X_μ in the expansion of θ^I on the Schubert basis of $H^*(\mathcal{F}_I)$. In other words,

$$c_\mu[\mathcal{F}_I] = \langle \theta^I, X_{\omega\mu} \rangle. \tag{17}$$

As in the case of the full flag variety $\mathcal{F}$, these scalar products can be computed with the help of the scalar product $(\ ,\ )$.

Let ω_I be the maximal element of $\mathfrak{S}_I$, and $\zeta_I := \omega_I \omega$. We have seen that

$$\square_\omega = \partial_\omega \theta = \partial_{\zeta_I} \partial_{\omega_I} \theta^I \theta_{J_1} \theta_{J_2} \cdots \theta_{J_r}.$$

The operator ∂_ω factorizes

$$\partial_\omega = \partial_{\zeta_i} \partial_{\omega_I}$$

so that

$$c_\mu[\mathcal{F}_I] = \partial_\omega \left(\theta^I X_{\omega\mu} \right)$$

$$= \partial_{\zeta_i} \partial_{\omega_i} \left(\theta^I X_{\omega\mu} \right) = \partial_{\zeta_i} \left(\theta^I \partial_{\omega_i} X_{\omega\mu} \right) \tag{18}$$

$$= \partial_{\zeta_I} \left(\theta^I X_{\omega\mu\omega_I} \right) = \partial_{\zeta_i} \left(\theta^I X_{\omega\mu\omega_I} \right) \left(\partial_{\omega_I} \theta_{J_1} \theta_{J_2} \cdots \theta_{J_r} / I! \right) \tag{19}$$

$$= \frac{1}{I!} \partial_\omega \left(\theta \, X_{\omega\mu\omega_I} \right) \tag{20}$$

$$= \frac{1}{I!} \left(1, X_{\omega\mu\omega_I} \right)$$

Equality (18) follows from the fact that θ^I is invariant under $\mathfrak{S}_I$ and thus commutes with ∂_{ω_I}. Now, θ is of degree $\binom{n}{2}$, and ∂_ω decreases degrees by $\ell(\omega) = \binom{n}{2}$. Thus $\partial_\omega(\theta)$

is a scalar which is checked to be $n!$. More generally, by direct product, one has for the maximal element of the Young subgroup $\mathfrak{S}_I$

$$\partial_{\omega_I}\theta_{J_1}\cdots\theta_{J_r} = I! := i_1!\cdots i_r!$$

and equality (19) follows from this identity.

Since θ^I as well as $X_{\omega\mu\omega_I}$ are invariant under $\mathfrak{S}_I$, they commute with ∂_{ω_I}, which is step (20).

Summarizing, we have the following expression for the components of the Chern class of $\mathcal{F}_I$ on the Schubert basis.

THEOREM 6.1. *Let $I = (i_1, \ldots, i_r)$ be a composition of n, $\mathfrak{S}_I$ and $\mathcal{F}_I$ the corresponding Young subgroup and flag variety. Let μ be a permutation which is minimum in its coset $\mu\mathfrak{S}_I$. Then, the Chern coefficient $c_\mu[\mathcal{F}_I]$ is given by*

$$c_\mu[\mathcal{F}_I] = (1, X_{\omega\mu\omega_I})/I!.$$

In particular, for the full flag variety (case $I = (1, 1, \ldots, 1)$), one has

$$c_\mu[\mathcal{F}] = (1, X_{\omega\mu}) = \square_\omega(X_{\omega\mu}) \tag{21}$$

and these numbers constitute the first column of the matrix $M(X, Z^\vee)$. Equivalently, they are equal to the absolute values of the entries of the last row of $M(Z^\vee, X)$.

In the case of a Grassmann manifold $G(p, p+q) = \mathcal{F}_{(p,q)}$, the basis of $H^*(\mathcal{F}_{(p,q)})$ consists of those X_μ for which $\mu_1 < \ldots < \mu_p$ and $\mu_{p+1} < \ldots < \mu_{p+q}$ (Grassmannian permutations). In fact, for such a permutation, X_μ is equal to the Schur function indexed by the partition $(\mu_1 - 1, \mu_2 - 2, \ldots, \mu_p - p)$ on the set of variables $\{x_1, \ldots, x_p\}$. Thus, the Chern coefficient $c_\mu[\mathcal{F}_{(p,q)}]$ is

$$c_\mu[\mathcal{F}_{(p,q)}] = \square_\omega\left(X_{(n+1-\mu_p,\ldots,n+1-\mu_1,n+1-\mu_n,\ldots,n+1-\mu_{p+1})}\right). \tag{22}$$

For example, up to a factor $(2!)^2$, the Chern coefficients of $\mathcal{F}_{(2,2)}$ are $4, 16, 28, 28, 48, 24$. They are given by the absolute values of the six entries of the bottom row of the matrix $M(Z^\vee, X)$ corresponding to columns indexed by permutations $\omega\mu$ where μ is Grassmannian.

7. Tables for $n = 4$.

7.1. *Affine Schubert polynomials.*

$Z_{4321} = x_1^3x_2^2x_3$

$Z_{4312} = x_1^3x_2^2$

$Z_{4231} = x_1^3x_2x_3$

$Z_{4213} = x_1^3x_2 \ -1/2\ x_1^3x_2x_3 \ - \ x_1^3x_2^2$

$Z_{4132} = x_1^3x_3 \ + \ x_1^3x_2 \ - \ x_1^3x_2x_3 \ -1/2\ x_1^3x_2^2$

$Z_{4123} = x_1^3$

$Z_{3421} = x_1^2x_2^2x_3$

$Z_{3412} = x_1^2x_2^2$

$Z_{3241} = x_1^2x_2x_3 \ -1/2\ x_1^2x_2^2x_3 \ - \ x_1^3x_2x_3$

$Z_{3214} = x_1^2x_2 \ -2/3\ x_1^2x_2x_3 \ -3/2\ x_1^2x_2^2 \ +1/3\ x_1^2x_2^2x_3 \ -2\ x_1^3x_2 \ +2/3\ x_1^3x_2x_3 \ + \ x_1^3x_2^2$

$Z_{3142} = x_1^2x_3 + x_1^2x_2 - x_1^2x_2x_3 - 2/3\, x_1^2x_2^2 - x_1^3x_3 - x_1^3x_2$

$Z_{3124} = x_1^2 - 1/2\, x_1^2x_3 - 1/2\, x_1^2x_2 + 1/2\, x_1^2x_2x_3 - 2\, x_1^3 + 1/2\, x_1^3x_3 + 1/2\, x_1^3x_2$

$Z_{2431} = x_1x_2^2x_3 + x_1^2x_2x_3 - x_1^2x_2^2x_3 - 1/2\, x_1^3x_2x_3$

$Z_{2413} = x_1x_2^2 - 1/2\, x_1x_2^2x_3 + x_1^2x_2 - 1/2\, x_1^2x_2x_3 - 2\, x_1^2x_2^2 + 1/2\, x_1^2x_2^2x_3 - 1/2\, x_1^3x_2 + 1/4\, x_1^3x_2x_3 + 1/2\, x_1^3x_2^2$

$Z_{2341} = x_1x_2x_3$

$Z_{2314} = x_1x_2 - 2/3\, x_1x_2x_3 - x_1x_2^2 - x_1^2x_2$

$Z_{2143} = x_1x_3 + x_1x_2 - 3/2\, x_1x_2x_3 - 2/3\, x_1x_2^2 + 1/3\, x_1x_2^2x_3 + x_1^2 - 2\, x_1^2x_3 - 8/3\, x_1^2x_2 + 7/3\, x_1^2x_2x_3 + 4/3\, x_1^2x_2^2 - 1/3\, x_1^2x_2^2x_3 - 3/2\, x_1^3 + 2\, x_1^3x_3 + 7/3\, x_1^3x_2 - 2/3\, x_1^3x_2x_3 - 1/3\, x_1^3x_2^2$

$Z_{2134} = x_1 - 1/2\, x_1x_2 + 1/2\, x_1x_2^2 - x_1^2 + 1/2\, x_1^2x_2 + x_1^3$

$Z_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 - 2\, x_1x_2^2x_3 + x_1^2x_3 + x_1^2x_2 - 2\, x_1^2x_2x_3 - 3/2\, x_1^2x_2^2 + x_1^2x_2^2x_3 - 2/3\, x_1^3x_3 - 2/3\, x_1^3x_2 + 2/3\, x_1^3x_2x_3 + 1/3\, x_1^3x_2^2$

$Z_{1423} = x_2^2 + x_1x_2 - x_1x_2^2 + x_1^2 - x_1^2x_2 - 2/3\, x_1^3$

$Z_{1342} = x_2x_3 + x_1x_3 + x_1x_2 - 2\, x_1x_2x_3 - 1/2\, x_1^2x_3 - 1/2\, x_1^2x_2 + 1/2\, x_1^2x_2x_3 + 1/2\, x_1^3x_3 + 1/2\, x_1^3x_2$

$Z_{1324} = x_2 - 1/2\, x_2x_3 - x_2^2 + x_1 - 1/2\, x_1x_3 - 5/2\, x_1x_2 + x_1x_2x_3 + 2\, x_1x_2^2 - 3/2\, x_1^2 + 1/4\, x_1^2x_3 + 9/4\, x_1^2x_2 - 1/4\, x_1^2x_2x_3 - 1/2\, x_1^2x_2^2 + x_1^3 - 1/4\, x_1^3x_3 - 1/4\, x_1^3x_2$

$Z_{1243} = x_3 + x_2 - x_2x_3 - 1/2\, x_2^2 + x_1 - x_1x_3 - 3/2\, x_1x_2 + x_1x_2x_3 + 1/2\, x_1x_2^2 - 1/2\, x_1^2 + 1/2\, x_1^2x_2$

$Z_{1234} = 1$

7.2. *Adjoint polynomials.*

$Z^\vee_{4321} = x_1^3x_2^2x_3$

$Z^\vee_{4312} = x_1^3x_2^2 - 2\, x_1^3x_2^2x_3$

$Z^\vee_{4231} = x_1^3x_2x_3 - 2\, x_1^3x_2^2x_3$

$Z^\vee_{4213} = x_1^3x_2 - 3/2\, x_1^3x_2x_3 - 3\, x_1^3x_2^2 + 3\, x_1^3x_2^2x_3$

$Z^\vee_{4132} = x_1^3x_3 + x_1^3x_2 - 3\, x_1^3x_2x_3 - 3/2\, x_1^3x_2^2 + 3\, x_1^3x_2^2x_3$

$Z^\vee_{4123} = x_1^3 - 2\, x_1^3x_3 - 4\, x_1^3x_2 + 6\, x_1^3x_2x_3 + 6\, x_1^3x_2^2 - 6\, x_1^3x_2^2x_3$

$Z^\vee_{3421} = x_1^2x_2^2x_3 - 2\, x_1^3x_2^2x_3$

$Z^\vee_{3412} = x_1^2x_2^2 - 2\, x_1^2x_2^2x_3 - 2\, x_1^3x_2^2 + 4\, x_1^3x_2^2x_3$

$Z^\vee_{3241} = x_1^2x_2x_3 - 3/2\, x_1^2x_2^2x_3 - 3\, x_1^3x_2x_3 + 3\, x_1^3x_2^2x_3$

$Z^\vee_{3214} = x_1^2x_2 - 4/3\, x_1^2x_2x_3 - 5/2\, x_1^2x_2^2 + 2\, x_1^2x_2^2x_3 - 4\, x_1^3x_2 + 4\, x_1^3x_2x_3 + 6\, x_1^3x_2^2 - 4\, x_1^3x_2^2x_3$

$Z^\vee_{3142} = x_1^2x_3 + x_1^2x_2 - 3\, x_1^2x_2x_3 - 4/3\, x_1^2x_2^2 + 8/3\, x_1^2x_2^2x_3 - 3\, x_1^3x_3 - 3\, x_1^3x_2 + 8\, x_1^3x_2x_3 + 8/3\, x_1^3x_2^2 - 16/3\, x_1^3x_2^2x_3$

$Z^\vee_{3124} = x_1^2 - 3/2\, x_1^2x_3 - 7/2\, x_1^2x_2 + 9/2\, x_1^2x_2x_3 + 5\, x_1^2x_2^2 - 4\, x_1^2x_2^2x_3 - 4\, x_1^3 + 9/2\, x_1^3x_3 + 25/2\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 12\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$

$Z^\vee_{2431} = x_1x_2^2x_3 + x_1^2x_2x_3 - 3\, x_1^2x_2^2x_3 - 3/2\, x_1^3x_2x_3 + 3\, x_1^3x_2^2x_3$

$$Z^\vee_{2413} = x_1x_2^2 - 3/2\, x_1x_2^2x_3 + x_1^2x_2 - 3/2\, x_1^2x_2x_3 - 4\, x_1^2x_2^2 + 9/2\, x_1^2x_2^2x_3 - 3/2\, x_1^3x_2 + 9/4\, x_1^3x_2x_3 + 9/2\, x_1^3x_2^2 - 9/2\, x_1^3x_2^2x_3$$

$$Z^\vee_{2341} = x_1x_2x_3 - 2\, x_1x_2^2x_3 - 4\, x_1^2x_2x_3 + 6\, x_1^2x_2^2x_3 + 6\, x_1^3x_2x_3 - 6\, x_1^3x_2^2x_3$$

$$Z^\vee_{2314} = x_1x_2 - 4/3\, x_1x_2x_3 - 3\, x_1x_2^2 + 8/3\, x_1x_2^2x_3 - 5\, x_1^2x_2 + 16/3\, x_1^2x_2x_3 + 10\, x_1^2x_2^2 - 8\, x_1^2x_2^2x_3 + 8\, x_1^3x_2 - 8\, x_1^3x_2x_3 - 12\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$

$$Z^\vee_{2143} = x_1x_3 + x_1x_2 - 5/2\, x_1x_2x_3 - 4/3\, x_1x_2^2 + 2\, x_1x_2^2x_3 + x_1^2 - 4\, x_1^2x_3 - 16/3\, x_1^2x_2 + 9\, x_1^2x_2x_3 + 16/3\, x_1^2x_2^2 - 6\, x_1^2x_2^2x_3 - 5/2\, x_1^3 + 7\, x_1^3x_3 + 9\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 6\, x_1^3x_2^2 + 6\, x_1^3x_2^2x_3$$

$$Z^\vee_{2134} = x_1 - 2\, x_1x_3 - 7/2\, x_1x_2 + 5\, x_1x_2x_3 + 9/2\, x_1x_2^2 - 4\, x_1x_2^2x_3 - 5\, x_1^2 + 8\, x_1^2x_3 + 31/2\, x_1^2x_2 - 18\, x_1^2x_2x_3 - 15\, x_1^2x_2^2 + 12\, x_1^2x_2^2x_3 + 11\, x_1^3 - 14\, x_1^3x_3 - 26\, x_1^3x_2 + 24\, x_1^3x_2x_3 + 18\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$

$$Z^\vee_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 - 4\, x_1x_2^2x_3 + x_1^2x_3 + x_1^2x_2 - 4\, x_1^2x_2x_3 - 5/2\, x_1^2x_2^2 + 6\, x_1^2x_2^2x_3 - 4/3\, x_1^3x_3 - 4/3\, x_1^3x_2 + 4\, x_1^3x_2x_3 + 2\, x_1^3x_2^2 - 4\, x_1^3x_2^2x_3$$

$$Z^\vee_{1423} = x_2^2 - 2\, x_2^2x_3 + x_1x_2 - 2\, x_1x_2x_3 - 5\, x_1x_2^2 + 8\, x_1x_2^2x_3 + x_1^2 - 2\, x_1^2x_3 - 5\, x_1^2x_2 + 8\, x_1^2x_2x_3 + 10\, x_1^2x_2^2 - 12\, x_1^2x_2^2x_3 - 4/3\, x_1^3 + 8/3\, x_1^3x_3 + 16/3\, x_1^3x_2 - 8\, x_1^3x_2x_3 - 8\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$

$$Z^\vee_{1342} = x_2x_3 - 2\, x_2^2x_3 + x_1x_3 + x_1x_2 - 6\, x_1x_2x_3 - 2\, x_1x_2^2 + 8\, x_1x_2^2x_3 - 7/2\, x_1^2x_3 - 7/2\, x_1^2x_2 + 25/2\, x_1^2x_2x_3 + 5\, x_1^2x_2^2 - 12\, x_1^2x_2^2x_3 + 9/2\, x_1^3x_3 + 9/2\, x_1^3x_2 - 12\, x_1^3x_2x_3 - 4\, x_1^3x_2^2 + 8\, x_1^3x_2^2x_3$$

$$Z^\vee_{1324} = x_2 - 3/2\, x_2x_3 - 3\, x_2^2 + 3\, x_2^2x_3 + x_1 - 3/2\, x_1x_3 - 15/2\, x_1x_2 + 9\, x_1x_2x_3 + 13\, x_1x_2^2 - 12\, x_1x_2^2x_3 - 9/2\, x_1^2 + 21/4\, x_1^2x_3 + 73/4\, x_1^2x_2 - 75/4\, x_1^2x_2x_3 - 22\, x_1^2x_2^2 + 18\, x_1^2x_2^2x_3 + 6\, x_1^3 - 27/4\, x_1^3x_3 - 75/4\, x_1^3x_2 + 18\, x_1^3x_2x_3 + 18\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$

$$Z^\vee_{1243} = x_3 + x_2 - 3\, x_2x_3 - 3/2\, x_2^2 + 3\, x_2^2x_3 + x_1 - 5\, x_1x_3 - 13/2\, x_1x_2 + 14\, x_1x_2x_3 + 15/2\, x_1x_2^2 - 12\, x_1x_2^2x_3 - 7/2\, x_1^2 + 11\, x_1^2x_3 + 31/2\, x_1^2x_2 - 26\, x_1^2x_2x_3 - 15\, x_1^2x_2^2 + 18\, x_1^2x_2^2x_3 + 5\, x_1^3 - 14\, x_1^3x_3 - 18\, x_1^3x_2 + 24\, x_1^3x_2x_3 + 12\, x_1^3x_2^2 - 12\, x_1^3x_2^2x_3$$

$$Z^\vee_{1234} = 1 - 2\, x_3 - 4\, x_2 + 6\, x_2x_3 + 6\, x_2^2 - 6\, x_2^2x_3 - 6\, x_1 + 10\, x_1x_3 + 22\, x_1x_2 - 28\, x_1x_2x_3 - 26\, x_1x_2^2 + 24\, x_1x_2^2x_3 + 16\, x_1^2 - 22\, x_1^2x_3 - 48\, x_1^2x_2 + 52\, x_1^2x_2x_3 + 44\, x_1^2x_2^2 - 36\, x_1^2x_2^2x_3 - 22\, x_1^3 + 28\, x_1^3x_3 + 52\, x_1^3x_2 - 48\, x_1^3x_2x_3 - 36\, x_1^3x_2^2 + 24\, x_1^3x_2^2x_3$$

7.3. *Transition matrices with Schubert polynomials.* The following matrices give the decompositions of the polynomials Z_μ and $Z^\vee_\mu$ in the basis of Schubert polynomials. Rows and columns are indexed by permutations in reverse lexicographic order:

[4321, 4312, 4231, 4213, 4132, 4123, 3421, 3412, 3241, 3214, 3142, 3124, 2431, 2413, 2341, 2314, 2143, 2134, 1432, 1423, 1342, 1324, 1243, 1234]

The bar over a number is to be interpreted as a minus sign.

7.3.1. $M(Z,X)$. The entry in row μ and column ν of the following matrix is equal to the coefficient of X_ν in Z_μ. This number is also the coefficient of $Z^\vee_{\omega\mu}$ in $X_{\omega\nu}$.

$$
\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{1} & \frac{\bar{1}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\bar{1}}{2} & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{1} & 0 & 0 & 0 & \frac{\bar{1}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{2}{3} & \bar{2} & 0 & 0 & \frac{1}{3} & \frac{\bar{3}}{2} & \frac{\bar{2}}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & \frac{\bar{2}}{3} & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \bar{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{\bar{1}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\bar{1}}{2} & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{4} & \frac{\bar{1}}{2} & 0 & 0 & \frac{1}{2} & \bar{2} & 0 & 0 & 0 & 0 & \frac{\bar{1}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & \frac{\bar{2}}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\bar{1}}{3} & \frac{\bar{2}}{3} & \frac{1}{3} & 2 & \frac{\bar{3}}{2} & \frac{\bar{1}}{3} & \frac{4}{3} & 2 & 0 & \bar{2} & 0 & \frac{1}{3} & \frac{\bar{2}}{3} & \frac{\bar{3}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & \frac{1}{2} & 0 & \frac{\bar{1}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{2}{3} & 0 & \frac{\bar{2}}{3} & 0 & 1 & \frac{\bar{3}}{2} & 0 & 0 & 0 & 0 & \bar{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\bar{2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\bar{1}}{2} & 0 & 0 & 0 & \bar{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\bar{1}}{4} & 1 & 0 & \frac{\bar{1}}{2} & \frac{\bar{1}}{4} & 0 & \frac{1}{4} & \frac{\bar{1}}{2} & 0 & 2 & 1 & \bar{1} & 0 & 0 & 0 & \bar{1} & \frac{\bar{1}}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & \frac{\bar{1}}{2} & \bar{1} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{matrix}
$$

7.3.2. $M(Z^\vee, X)$. The entry in row μ and column ν of the following matrix is equal to the coefficient of X_ν in $Z^\vee_\mu$. This number is also the coefficient of $Z_{\omega\mu}$ in $X_{\omega\nu}$.

$$
\begin{array}{cccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \overline{3} & \frac{\overline{3}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \frac{\overline{3}}{2} & \overline{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{6} & 6 & 6 & \overline{2} & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & \overline{2} & 0 & 0 & 0 & 0 & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & \overline{3} & 0 & 0 & 0 & \frac{\overline{3}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{4} & 6 & 4 & \overline{4} & 0 & 0 & 2 & \frac{\overline{5}}{2} & \frac{\overline{4}}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\overline{16}}{3} & \frac{8}{3} & 8 & 0 & \overline{3} & 0 & \frac{8}{3} & \frac{\overline{4}}{3} & \overline{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{12} & \overline{12} & 8 & \frac{9}{2} & \overline{4} & \overline{4} & 5 & \frac{9}{2} & \overline{2} & \frac{\overline{3}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & \frac{\overline{3}}{2} & 0 & 0 & 0 & \overline{3} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\overline{9}}{2} & \frac{9}{2} & \frac{9}{4} & \frac{\overline{3}}{2} & 0 & 0 & \frac{9}{2} & \overline{4} & 0 & 0 & 0 & 0 & \frac{\overline{3}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{6} & 0 & 6 & 0 & 0 & 0 & 6 & 0 & \overline{2} & 0 & 0 & 0 & \overline{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{12} & \overline{8} & 8 & 0 & 0 & \overline{8} & 10 & \frac{8}{3} & \overline{2} & 0 & 0 & \frac{8}{3} & \overline{3} & \frac{\overline{4}}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & \overline{6} & \overline{12} & 2 & 7 & \frac{\overline{5}}{2} & \overline{6} & \frac{16}{3} & 7 & 0 & \overline{4} & 0 & 2 & \frac{\overline{4}}{3} & \frac{\overline{5}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{12} & 18 & 24 & \overline{12} & \overline{14} & 11 & 12 & \overline{15} & \overline{14} & 3 & 8 & \overline{3} & \overline{4} & \frac{9}{2} & 5 & \frac{\overline{3}}{2} & \overline{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{4} & 2 & 4 & 0 & \frac{\overline{4}}{3} & 0 & 6 & \frac{\overline{5}}{2} & 0 & 0 & 0 & 0 & \overline{4} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
8 & \overline{8} & \overline{8} & \frac{8}{3} & \frac{8}{3} & \frac{\overline{4}}{3} & \overline{12} & 10 & 0 & 0 & 0 & 0 & 8 & \overline{3} & 0 & 0 & 0 & 0 & \overline{2} & 1 & 0 & 0 & 0 & 0 \\
8 & \overline{4} & \overline{12} & 0 & \frac{9}{2} & 0 & \overline{12} & 5 & \frac{9}{2} & 0 & \frac{\overline{3}}{2} & 0 & 8 & 0 & \overline{4} & 0 & 0 & 0 & \overline{2} & 0 & 1 & 0 & 0 & 0 \\
\overline{12} & 18 & 18 & \overline{12} & \frac{\overline{27}}{4} & 6 & 18 & \overline{22} & \frac{\overline{27}}{4} & 3 & \frac{9}{4} & \frac{\overline{3}}{2} & \overline{12} & 10 & 6 & \overline{3} & 0 & 0 & 3 & \overline{3} & \frac{\overline{3}}{2} & 1 & 0 & 0 \\
\overline{12} & 12 & 24 & \overline{4} & \overline{14} & 5 & 18 & \overline{15} & \overline{14} & 0 & 8 & 0 & \overline{12} & \frac{9}{2} & 11 & 0 & \overline{2} & 0 & 3 & \frac{\overline{3}}{2} & \overline{3} & 0 & 1 & 0 \\
24 & \overline{36} & \overline{48} & 24 & 28 & \overline{22} & \overline{36} & 44 & 28 & \overline{6} & \overline{16} & 6 & 24 & \overline{20} & \overline{22} & 6 & 4 & \overline{2} & \overline{6} & 6 & 6 & \overline{2} & \overline{2} & 1
\end{array}
$$

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