Bootstrap Confidence Intervals

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- Expanded percentile
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- Bias-corrected & accelerated
- Examples (revisited)

For a thorough treatment see:

Hesterberg, Tim (2014). What teachers should know about the bootstrap: Resampling in the undergraduate statistics curriculum. arXiv:1411.5279v1.

Confidence Intervals

Classic Confidence Interval Formula

A symmetric $100(1-\alpha)\%$ confidence interval (CI) has the form:

$$\hat{\theta} \pm t_{\alpha/2}\sigma_{\hat{\theta}}$$

where $\hat{\theta}$ is our estimate of θ, $\sigma_{\hat{\theta}}$ is the standard error of $\hat{\theta}$, and $t_{\alpha/2}$ is the critical value of the test statistic, i.e., $P(t \leq t_{\alpha/2}) = \alpha/2$.

- Assumes that distribution of test statistic is symmetric around zero
- As $n \to \infty$ we often have $\hat{\theta} \dot{\sim} \mathrm{N}(\theta, \sigma^2_{\hat{\theta}})$, so that $t_{\alpha/2} = z_{\alpha/2}$.

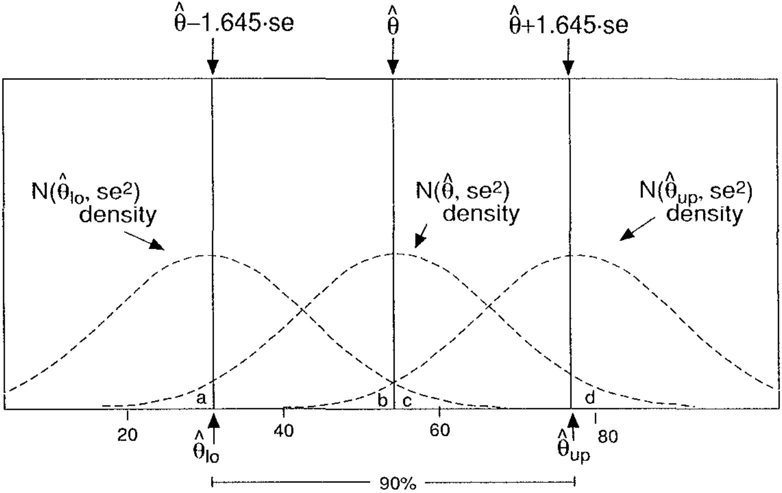
More generally we can write a $100(1-\alpha)\%$ CI as

$$[\hat{\theta} - t_{1-\alpha/2}\sigma_{\hat{\theta}},\ \hat{\theta} - t_{\alpha/2}\sigma_{\hat{\theta}}]$$

where $P(t \leq t_{1-\alpha/2}) = 1 - \alpha/2$ and $P(t \leq t_{\alpha/2}) = \alpha/2$

Visualization of 90% Gaussian Confidence Intervals

Figure 12.1: *An Introduction to the Bootstrap* (Efron & Tibshirani, 1993)



Proper Interpretation of CIs

Unfortunately, (frequentist) confidence intervals don't have the interpretation that one might expect (or hope) for...

- Incorrect interpretations of CIs are prevalent in scientific papers

Interpreting a 99% Confidence Interval:

- *Correct*: through repeated samples, e.g., 99 out of 100 confidence intervals would be expected to contain true θ with $\alpha = .01$
- *Wrong*: through one sample; e.g., there is a 99% chance the confidence interval around my $\hat{\theta}$ contains the true θ (with $\alpha = .01$)

Proper Interpretation of CIs: Example

```
> set.seed(1)
> n = 100
> B = 10^4
> X = replicate(B, rnorm(n))
> xbar = apply(X, 2, mean)
> xsd = apply(X, 2, sd)
> cilo = xbar - qt(.95, df=n-1)*(xsd/sqrt(n))
> ciup = xbar - qt(.05, df=n-1)*(xsd/sqrt(n))
> ci90 = (0>=cilo & 0<=ciup)
> mean(ci90)
[1] 0.902
> summary(ci90)
   Mode   FALSE    TRUE    NA's
logical     980    9020       0
```

Some Properties of Confidence Intervals

Two properties we can use to describe a confidence interval:

- length $= \hat{\theta}_{\text{up}} - \hat{\theta}_{\text{lo}}$
- shape $= \frac{\hat{\theta}_{\text{up}} - \hat{\theta}}{\hat{\theta} - \hat{\theta}_{\text{lo}}}$

Note that. . .

- Length: describes the overall size of the CI
- Shape: describes the asymmetry of the CI

shape > 1 indicates a greater distance between $\hat{\theta}_{\text{up}}$ and $\hat{\theta}$ than between $\hat{\theta}_{\text{lo}}$ and $\hat{\theta}$

Defining a Good Confidence Interval

What is a “good” bootstrap confidence interval?

- If an exact CI can be formed (e.g., sample mean), bootstrap CI should closely match exact CI
- If an exact CI cannot be formed (e.g., sample median), bootstrap CI should give accurate coverage probabilities

Note that a narrower CI is not necessarily a better CI. Length and shape are only imporant if the coverage probabilities are accurate.

Different bootstrap CI methods have different coverage accuracies.

First and Second Order Accurate

"Big Oh" notation: $f(x) = O(g(x))$ is read as "$f(x)$ is big-oh of $g(x)$"

- $f(x) = O(g(x))$ as $x \to \infty$ if and only if $|f(x)| \leq h|g(x)|$ for all $x \geq x_0$ and some $h > 0$
- For sufficiently large x, the magnitude of $f(x)$ is at most h times the magnitude of $g(x)$

A confidence interval is first-order accurate if the non-coverage probability on each side differs from the nominal value by $O(n^{-1/2})$.

- $P(\theta < \hat{\theta}_{\mathrm{lo}}) = \alpha + h_{\mathrm{lo}}/\sqrt{n}$ and $P(\theta > \hat{\theta}_{\mathrm{up}}) = \alpha + h_{\mathrm{up}}/\sqrt{n}$

A confidence interval is second-order accurate if the non-coverage probability on each side differs from the nominal value by $O(n^{-1})$.

- $P(\theta < \hat{\theta}_{\mathrm{lo}}) = \alpha + h_{\mathrm{lo}}/n$ and $P(\theta > \hat{\theta}_{\mathrm{up}}) = \alpha + h_{\mathrm{up}}/n$

Basic Bootstrap CIs

t Confidence Interval with Bootstrap Standard Error

Uses the classic CI formula with the bootstrap SE estimate:

$$\text{Classic SE}: \quad \hat{\theta} \pm t_{\alpha/2}\hat{\sigma}_{\hat{\theta}}$$
$$\text{Bootstrap SE}: \quad \hat{\theta} \pm t_{\alpha/2}\hat{\sigma}_{B}$$

No real benefit over the class t interval using $\hat{\sigma}_{\hat{\theta}}$.

This CI procedure is only first-order accurate.

Properties of t CI with Bootstrap SE

Pros:

- Simple to form and easy to understand
- Can be applied to situations where $\sigma_{\hat{\theta}}$ is difficult to derive

Cons:

- Tends to be too narrow for small n because $\hat{\sigma}_B$ is too narrow.
- Comparable to using the MLE $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$ instead of the unbiased estimate $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
- Can perform poorly if distribution is highly skewed

Bootstrap Confidence Intervals via Percentiles

Another intuitive approach is to use the 100α-th and $100(1-\alpha)$-th percentiles of bootstrap distribution of $\hat{\theta}$.

For example, if we have $B = 10,000$ bootstrap replications of $\hat{\theta}$

$$\hat{\theta}^*_{(1)} \leq \hat{\theta}^*_{(2)} \leq \cdots \leq \hat{\theta}^*_{(B)}$$

we would define the 90% confidence interval using

$$[\hat{\theta}^*_{(500)}, \hat{\theta}^*_{(9500)}] = [\hat{\theta}_{\text{lo}}, \hat{\theta}_{\text{up}}]$$

This CI procedure is only first-order accurate.

Properties of Bootstrap Percentile CIs

Pros:

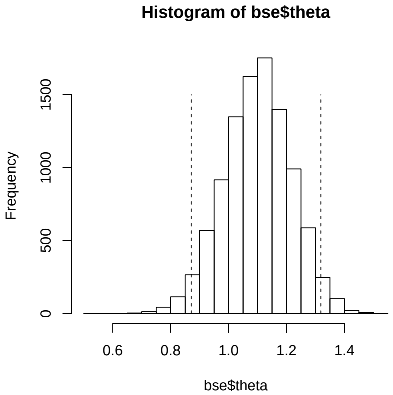
- Simple to form and easy to understand
- Range preserving and transformation invariant
- Advantage over t CI with bootstrap SE when data are skewed

Cons:

- Tends to be too narrow for small n (worse than t w/ bootstrap SE)
- Comparable to using $z_{\alpha/2}\hat{\sigma}/\sqrt{n}$ instead of $t_{\alpha/2}s/\sqrt{n}$
- Does partial skewness correction, which adds random variability

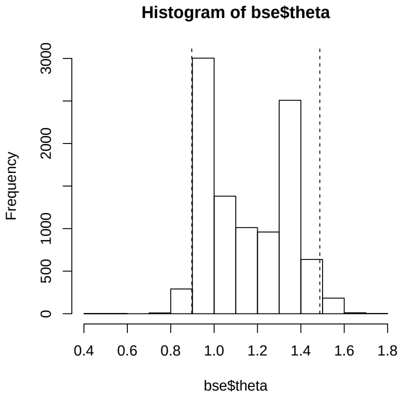
Example 1: Sample Mean CI

```
> dev.new(width=5,height=5,noRStudioGD=TRUE)
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,mean)
> mean(x)
[1] 1.100448
> c(mean(x)-qt(0.975,df=n-1)*sd(x)/sqrt(n),
+   mean(x)-qt(0.025,df=n-1)*sd(x)/sqrt(n))
[1] 0.8641687 1.3367278
> ci = quantile(bse$theta,c(0.025,0.975))
> ci
     2.5%     97.5%
0.8707902 1.3187369
> hist(bse$theta)
> lines(rep(ci[1],2),c(0,1500),lty=2)
> lines(rep(ci[2],2),c(0,1500),lty=2)
```



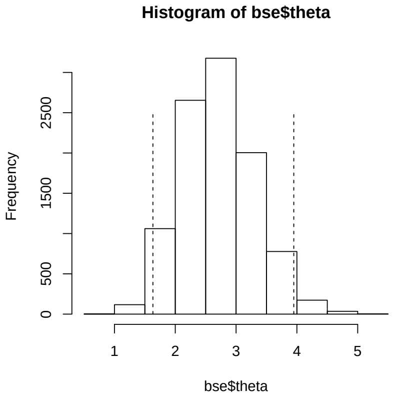
Example 2: Sample Median CI

```
> dev.new(width=5,height=5,noRStudioGD=TRUE)
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,median)
> median(x)
[1] 1.129104
> ci = quantile(bse$theta,c(0.025,0.975))
> ci
     2.5%     97.5%
0.8972123 1.4874291
> hist(bse$theta)
> lines(rep(ci[1],2),c(0,4000),lty=2)
> lines(rep(ci[2],2),c(0,4000),lty=2)
```



Example 3: Sample Variance CI

```
> dev.new(width=5,height=5,noRStudioGD=TRUE)
> set.seed(1)
> n = 50
> x = rnorm(n,sd=2)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,var)
> var(x)
[1] 2.764863
> c((n-1)*var(x)/qchisq(0.975,df=n-1),
+   (n-1)*var(x)/qchisq(0.025,df=n-1))
[1] 1.929274 4.293414
> ci = quantile(bse$theta,c(0.025,0.975))
> ci
    2.5%    97.5%
1.632342 3.948860
> hist(bse$theta)
> lines(rep(ci[1],2),c(0,2500),lty=2)
> lines(rep(ci[2],2),c(0,2500),lty=2)
```



Transformation Respecting Property of Percentile CIs

```
> set.seed(1)
> x = rnorm(50,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,function(x) exp(mean(x)))
> exp(mean(x))
[1] 3.005513
> mean(x)
[1] 1.100448
> ci = quantile(bse$theta,c(0.025,0.975))
> ci
    2.5%    97.5%
2.388798 3.738696
> bse = bootse(bsamp,mean)
> quantile(bse$theta,c(0.025,0.975))
     2.5%     97.5%
0.8707902 1.3187369
> log(ci)
     2.5%     97.5%
0.8707902 1.3187369
```

Better Bootstrap CIs

Expanded Percentile Confidence Intervals

Can interpret t interval as multiplying the length of a normal interval

$$\bar{x} \pm z_{\alpha/2}\hat{\sigma}/\sqrt{n}$$

by a factor $a_{\alpha,n} = (t_{\alpha/2}/z_{\alpha/2})(s/\hat{\sigma})$ where

- $s = \{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\}^{1/2}$
- $\hat{\sigma} = \{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\}^{1/2}$

Percentile CIs comparable to using $z_{\alpha/2}\hat{\sigma}/\sqrt{n}$ instead of $t_{\alpha/2}s/\sqrt{n}$, so we can use an adjustment to correct for narrowness bias.

- Don't want to apply correction by multiplying both sides of interval by $a_{\alpha,n}$ because this would not be transformation invariant
- Instead, apply correction to quantiles of bootstrap distribution

Expanded Percentile Confidence Intervals (continued)

If the bootstrap distribution is approximately normal, then

$$\hat{F}^{-1}(\alpha/2) \approx \hat{\theta} + z_{\alpha/2}\hat{\sigma}/\sqrt{n}$$

and we want to find a modified quantile value α' such that

$$\begin{aligned}\hat{F}^{-1}(\alpha'/2) &\approx \hat{\theta} + z_{\alpha'/2}\hat{\sigma}/\sqrt{n} \\ &= \hat{\theta} + t_{\alpha/2}s/\sqrt{n}\end{aligned}$$

This implies that $z_{\alpha'/2} = \sqrt{n/(n-1)}t_{\alpha/2}$ so the modified quantile is

$$\alpha'/2 = \Phi(\sqrt{n/(n-1)}t_{\alpha/2})$$

Properties of Expanded Percentile CIs

Pros:

- Simple to form and easy to understand
- Range preserving and transformation invariant
- Corrects for narrowness bias of percentile CIs

Cons:

- Does partial skewness correction, which adds random variability
- No correction for bias, and doesn't fully correct for skewness
- Only first-order accurate

Example 1: Sample Mean CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,mean)
> mean(x)
[1] 1.100448
> c(mean(x)-qnorm(0.975)*sd(x)*sqrt((n-1)/n)/sqrt(n),
+   mean(x)-qnorm(0.025)*sd(x)*sqrt((n-1)/n)/sqrt(n))
[1] 0.872318 1.328579
> quantile(bse$theta,c(0.025,0.975))
     2.5%     97.5%
0.8707902 1.3187369
> alphaD2 = pnorm(sqrt(n/(n-1))*qt(.025,df=n-1))
> alphaD2
[1] 0.02117941
> c(mean(x)-qt(0.975,df=n-1)*sd(x)/sqrt(n),
+   mean(x)-qt(0.025,df=n-1)*sd(x)/sqrt(n))
[1] 0.8641687 1.3367278
> quantile(bse$theta,c(alphaD2,1-alphaD2))
2.117941% 97.88206%
 0.862373  1.326751
```

Example 2: Sample Median CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,median)
> median(x)
[1] 1.129104
> quantile(bse$theta,c(0.025,0.975))
     2.5%     97.5%
0.8972123 1.4874291
> alphaD2 = pnorm(sqrt(n/(n-1))*qt(.025,df=n-1))
> alphaD2
[1] 0.02117941
> quantile(bse$theta,c(alphaD2,1-alphaD2))
2.117941% 97.88206%
 0.892433  1.487429
```

Example 3: Sample Variance CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,sd=2)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,var)
> var(x)
[1] 2.764863
> quantile(bse$theta,c(0.025,0.975))
    2.5%    97.5%
1.632342 3.948860
> alphaD2 = pnorm(sqrt(n/(n-1))*qt(.025,df=n-1))
> alphaD2
[1] 0.02117941
> quantile(bse$theta,c(alphaD2,1-alphaD2))
2.117941% 97.88206%
 1.600105  3.997069
```

Bootstrap t-Table Confidence Intervals

Given B bootstrap samples $\mathbf{x}_1^*, \ldots, \mathbf{x}_B^*$, we compute

$$t_b^* = \frac{\hat{\theta}_b^* - \hat{\theta}}{\hat{\sigma}_b}$$

where $\hat{\sigma}_b$ is standard error for b-th bootstrap sample.

Note that $\hat{\sigma}_b$ may not have a closed form solution:

- If $\hat{\theta}$ is sample mean, then $\hat{\sigma}_b = \{\sum_{i=1}^n (x_{i(b)}^* - \bar{x}_b^*)^2 / n^2\}^{1/2}$
- For other statistics, need bootstrap SE for each bootstrap sample

Bootstrap t-Table Confidence Intervals (continued)

Given t_b^* for $b \in \{1, \ldots, B\}$, define the α-th quantile q_α as

$$\#\{t_b^* \leq q_\alpha\}/B = \alpha$$

and note that we have

$$\begin{aligned} 1 - \alpha &= P(q_{\alpha/2} < t_b^* < q_{1-\alpha/2}) \\ &\approx P(q_{\alpha/2} < t < q_{1-\alpha/2}) \\ &= P(q_{\alpha/2}\hat{\sigma}_B < \hat{\theta} - \theta < q_{1-\alpha/2}\hat{\sigma}_B) \\ &= P(\hat{\theta} - q_{\alpha/2}\hat{\sigma}_B > \theta > \hat{\theta} - q_{1-\alpha/2}\hat{\sigma}_B) \end{aligned}$$

Form the "bootstrap-t" interval: $[\hat{\theta} - q_{1-\alpha/2}\hat{\sigma}_B, \ \hat{\theta} - q_{\alpha/2}\hat{\sigma}_B]$

Properties of Bootstrap t-Table CIs

Pros:

- Simple idea with intuitive procedure
- Works well for location parameters
- Second-order accurate

Cons:

- Not range preserving or transformation invariant
- Formation of $\hat{\sigma}_b$ requires *iterated bootstrap*
- Doesn't work as well for correlation/association measures

Example 1: Sample Mean CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,mean)
> bsampSE = apply(bsamp, 2, sd) * sqrt((n-1)/n) / sqrt(n)
> theta = mean(x)
> Z = (bse$theta - theta) / bsampSE
> cval = quantile(Z, probs=c(0.025,0.975))

# bootstrap t-table:
> c(theta - cval[2]*bse$se, theta - cval[1]*bse$se)
    97.5%      2.5%
0.8538653 1.3189295

# percentile:
> quantile(bse$theta,c(0.025,0.975))
     2.5%     97.5%
0.8707902 1.3187369
```

Example 2: Sample Median CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,median)
> theta = median(x)
> bsampSE = rep(0, ncol(bsamp))
> for(k in 1:ncol(bsamp)){
+  bsampSE[k] = bootse(bootsamp(bsamp[,k],nsamp=2000),median)$se
+ }
> Z = (bse$theta - theta) / bsampSE
> cval = quantile(Z, probs=c(0.025,0.975))

# bootstrap t-table
> c(theta - cval[2]*bse$se, theta - cval[1]*bse$se)
    97.5%      2.5%
0.6226602 1.5705624

# percentile
> quantile(bse$theta,c(0.025,0.975))
     2.5%     97.5%
0.8972123 1.4874291
```

Example 3: Sample Variance CI (revisited)

```
> set.seed(1)
> n = 50
> x = rnorm(n,sd=2)
> bsamp = bootsamp(x)
> bse = bootse(bsamp,var)
> theta = var(x)
> bsampSE = rep(0, ncol(bsamp))
> for(k in 1:ncol(bsamp)){
+  bsampSE[k] = bootse(bootsamp(bsamp[,k],nsamp=2000),var)$se
+ }
> Z = (bse$theta - theta) / bsampSE
> cval = quantile(Z, probs=c(0.025,0.975))

# bootstrap t-table
> c(theta - cval[2]*bse$se, theta - cval[1]*bse$se)
   97.5%      2.5%
1.796957 4.971694

# percentile
> quantile(bse$theta,c(0.025,0.975))
    2.5%     97.5%
1.632342 3.948860
```

BC_a Bootstrap CIs

BC_a intervals use percentiles of bootstrap distribution, but they do not necessarily use the 100α-th and 100$(1-\alpha)$-th percentiles.

- Depend on acceleration parameter $\hat{a}$
- Depend on bias-correction factor $\hat{z}_0$

BC_a intervals have the form: $[\hat{\theta}^*_{(\alpha_1)}, \hat{\theta}^*_{(\alpha_2)}] = [\hat{\theta}_{\mathrm{lo}}, \hat{\theta}_{\mathrm{up}}]$

- $\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{(\alpha)}}{1 - \hat{a}(\hat{z}_0 + z_{(\alpha)})}\right)$
- $\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{(1-\alpha)}}{1 - \hat{a}(\hat{z}_0 + z_{(1-\alpha)})}\right)$
- $\Phi(\cdot)$ is the cdf of standard normal (`pnorm`)
- $z_{(\alpha)}$ is the 100α-th percentile of standard normal

Estimating Bias-Correction Factor

Note that if $\hat{a} = \hat{z}_0 = 0$, then...

- $\alpha_1 = \Phi(z_{(\alpha)}) = \alpha$
- $\alpha_2 = \Phi(z_{(1-\alpha)}) = 1 - \alpha$

and the BC$_a$ interval is the same as the percentile interval.

The bias-correction factor is estimated as

$$\hat{z}_0 = \Phi^{-1}\left(\#\{\hat{\theta}_b^* < \hat{\theta}\}/B\right)$$

where $\Phi^{-1}(\cdot)$ is the inverse cdf of standard normal (`qnorm`).

- $\hat{z}_0$ measures median bias of $\hat{\theta}^*$, i.e., difference between $\mathrm{median}(\hat{\theta}_b^*)$ and $\hat{\theta}$

Estimating Acceleration Factor

The acceleration factor can be calculated using a jackknife approach:

- Reminder: $\hat{\theta}_{(i)} = s(\mathbf{x}_{(i)})$ is estimate of θ holding out x_i
- Reminder: $\hat{\theta}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(i)}$ is the average of jackknife estimates

The acceleration factor can be expressed as

$$\hat{a} = \frac{\sum_{i=1}^{n}(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^3}{6\{\sum_{i=1}^{n}(\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^2\}^{3/2}}$$

which estimates the rate of change of the standard error of $\hat{\theta}$ with respect to the true parameter θ.

- Assuming that $\hat{\theta} \dot{\sim} \mathrm{N}(\theta, \sigma_{\hat{\theta}}^2)$ assumes that $\sigma_{\hat{\theta}}$ is the same for all θ
- $\hat{a}$ corrects for this (possibly unrealistic) assumption

Properties of Bias-Corrected and Accelerated CIs

Pros:

- Range preserving and transformation invariant
- Works well for a variety parameters
- Second-order accurate

Cons:

- Requires estimation of acceleration and bias-correction
- Less intuitive than other methods

BC_a Confidence Intervals in R

We could easily write our own BC_a confidence interval function using the bootstrap and jackknife functions we've already created.

- We can calculate $\hat{z}_0$ using output of `bootse`
- We can calculate $\hat{a}$ using output of `jackse`

But there is already the `bcanon` function (in `bootstrap` package).

- Takes in `x`, `nboot`, and `theta` as necessary inputs
- Outputs `confpoint`, `z0`, `acc`, and `u` (jackknife influence)

BC_a Confidence Intervals in R (continued)

Our BC_a confidence interval function would look something like…

```
bcafun <- function(x,nboot,theta,...,alpha=0.05){
  theta.hat = theta(x)
  nx = length(x)
  bse = bootse(bootsamp(x,nboot),theta,...)
  jse = jackse(jacksamp(x),theta,...)
  z0 = qnorm(sum(bse$theta<theta.hat)/nboot)
  atop = sum((mean(jse$theta)-jse$theta)^3)
  abot = 6*(((jse$se^2)*nx/(nx-1))^(3/2))
  ahat = atop/abot
  alpha1 = pnorm(z0+(z0+qnorm(alpha))/(1-ahat*(z0+qnorm(alpha))))
  alpha2 = pnorm(z0+(z0+qnorm(1-alpha))/(1-ahat*(z0+qnorm(1-alpha))))
  confpoint = quantile(bse$theta,probs=c(alpha1,alpha2))
  list(confpoint=confpoint,z0=z0,acc=ahat,u=(jse$theta-theta.hat),
       theta=bse$theta,se=bse$se)
}
```

Note that this forms a $100(1-2\alpha)\%$ confidence interval.

Example 1: Sample Mean (revisited)

```
> library(bootstrap)
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> c(mean(x)-qt(0.975,df=n-1)*sd(x)/sqrt(n),
+   mean(x)-qt(0.025,df=n-1)*sd(x)/sqrt(n))
[1] 0.8641687 1.3367278
> mybca = bcafun(x,10000,mean,alpha=0.025)
> quantile(mybca$theta,probs=c(0.025,0.975))
     2.5%     97.5%
0.8707902 1.3187369
> mybca$conf
1.933122%  96.8455%
0.8573764 1.3077089
> bca = bcanon(x,10000,mean,alpha=c(0.025,0.975))
> bca$conf
     alpha bca point
[1,] 0.025 0.8582165
[2,] 0.975 1.3118539
```

Example 2: Sample Median (revisited)

```
> library(bootstrap)
> set.seed(1)
> n = 50
> x = rnorm(n,mean=1)
> mybca = bcafun(x,10000,median,alpha=0.025)
> quantile(mybca$theta,c(0.025,0.975))
     2.5%     97.5%
0.8972123 1.4874291
> mybca$conf
1.710689% 96.42576%
 0.892433  1.452685
> bca = bcanon(x,10000,median,alpha=c(0.025,0.975))
> bca$conf
     alpha bca point
[1,] 0.025 0.8880815
[2,] 0.975 1.4732532
```

Example 3: Sample Variance (revisited)

```
> library(bootstrap)
> set.seed(1)
> n = 50
> x = rnorm(n,sd=2)
> mybca = bcafun(x,10000,var,alpha=0.025)
> quantile(mybca$theta,c(0.025,0.975))
    2.5%    97.5%
1.632342 3.948860
> mybca$conf
7.083711% 99.53268%
 1.864701  4.409839
> bca = bcanon(x,10000,var,alpha=c(0.025,0.975))
> bca$conf
     alpha bca point
[1,] 0.025  1.863058
[2,] 0.975  4.389655
```

Example 1: Sample Mean Results Summary

```
> tab.mean = rbind(standZ,standT,prcnt.mean,
+                  eprcnt.mean,bootT.mean,bca.mean)
> rownames(tab.mean) = c("standZ","standT","prcnt",
+                         "eprcnt","bootT","bca")
> round(tab.mean,4)
         2.5%  97.5%
standZ 0.8723 1.3286
standT 0.8642 1.3367
prcnt  0.8708 1.3187
eprcnt 0.8624 1.3268
bootT  0.8539 1.3189
bca    0.8574 1.3077
```

Example 2: Sample Median Results Summary

```
> tab.med = rbind(prcnt.med,eprcnt.med,
+                 bootT.med,bca.med)
> rownames(tab.med) = c("prcnt","eprcnt",
+                        "bootT","bca")
> round(tab.med,4)
         2.5%  97.5%
prcnt  0.8972 1.4874
eprcnt 0.8924 1.4874
bootT  0.6227 1.5706
bca    0.8924 1.4527
```

Example 3: Sample Variance Results Summary

```
> tab.var = rbind(standZ.var,prcnt.var,
+                 eprcnt.var,bootT.var,bca.var)
> rownames(tab.var) = c("standZ","prcnt",
+                       "eprcnt","bootT","bca")
> round(tab.var,4)
         2.5%  97.5%
standZ 1.9293 4.2934
prcnt  1.6323 3.9489
eprcnt 1.6001 3.9971
bootT  1.7970 4.9717
bca    1.8647 4.4098
```